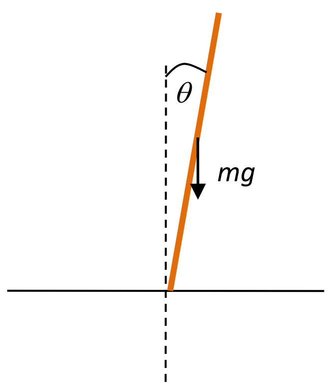
# Teacher notes
# Topic E

## Heisenberg's pencil.

Suppose we want to balance a pencil vertically on a horizontal surface. Heisenberg's uncertainty principle says that there will always be an uncertainty in the angle to the vertical (so the pencil will fall) and there will be an uncertainty in the angular speed of the pencil so again it will fall. So, what does Physics say about the maximum possible time for which we can keep the pencil almost vertical?




The torque provided by the weight about the point where the pencil touches the ground is $mg\frac{L}{2}\sin\theta$ and so $mg\frac{L}{2}\sin\theta = I\alpha$. The moment of inertia is $\frac{1}{3}mL^2$ and so

$\alpha = \frac{3g}{2L}\sin\theta$ or $\frac{d^2\theta}{dt^2} = \frac{3g}{2L}\sin\theta$.

Assume that $\theta$ is small so that $\sin\theta \approx \theta$, then

$$\frac{d^2\theta}{dt^2} - \frac{3g}{2L}\theta \approx 0$$

The solution is

$$\theta(t) = Ae^{\frac{t}{\tau}} + Be^{-\frac{t}{\tau}}$$

where $\tau = \sqrt{\frac{3L}{2g}}$.

The initial conditions are: $\theta(0) = \theta_0$ and $\frac{d\theta(0)}{dt} = \omega_0$. We are assuming that no hard how you try there will always be an uncertainty in the value of $\theta$ making it non-zero and similarly, you can never avoid giving the pencil some initial angular speed. In either case, i.e. $\theta_0 \neq 0$ or $\omega_0 \neq 0$, the pencil will topple over and fall to the ground.

Implementing the initial conditions, we find

$$\theta_0 = A + B$$

and

$$\omega_0 = \frac{A}{\tau} - \frac{B}{\tau} \text{ or } \omega_0 \tau = A - B$$

This implies that

$$A = \frac{1}{2}(\theta_0 + \omega_0 \tau) \text{ and } B = \frac{1}{2}(\theta_0 - \omega_0 \tau) \text{ so that}$$

$$\theta(t) = \frac{1}{2}(\theta_0 + \omega_0 \tau)e^{\frac{t}{\tau}} + \frac{1}{2}(\theta_0 - \omega_0 \tau)e^{-\frac{t}{\tau}}$$

The negative exponential will become negligible quickly and we may ignore it. Then $\theta(t) \approx \frac{1}{2}(\theta_0 + \omega_0 \tau)e^{\frac{t}{\tau}}$.

Heisenberg's uncertainty relation for angular momentum says that $\Delta L \Delta \theta \geq \frac{\hbar}{2}$ i.e.,

$\frac{1}{3} m L^2 \omega_0 \theta_0 \geq \frac{\hbar}{2} \Rightarrow \omega_0 \geq \frac{3\hbar}{2mL^2 \theta_0}$. Then

$$\theta(t) \geq \frac{1}{2}(\theta_0 + \frac{3\hbar}{2mL^2 \theta_0} \tau)e^{\frac{t}{\tau}}$$

We want to balance the pencil, in other words we want to keep $\theta(t)$ small for as long as possible. So, we want to find the value of $\theta_0$ that minimizes $(\theta_0 + \omega_0 \tau)$. We take the derivative of the bracket to get

$\frac{d}{d\theta_0}(\theta_0 + \frac{3\hbar}{2mL^2 \theta_0} \tau) = 1 - \frac{3\hbar}{2mL^2 \theta_0^2} \tau = 0 \Rightarrow \theta_0^2 = \frac{3\hbar \tau}{2mL^2}$. Hence

$$\theta(t) \geq \frac{1}{2}\left(\sqrt{\frac{3\hbar\tau}{2mL^2}} + \frac{3\hbar\tau}{mL^2\sqrt{\frac{3\hbar\tau}{2mL^2}}}\right)e^{\frac{t}{\tau}} = \sqrt{\frac{3\hbar\tau}{2mL^2}}e^{\frac{t}{\tau}}.$$

We may estimate that the limit of what small angle means is about $\theta(t) = \frac{1}{2}$, so we get

$\frac{1}{2} \geq \frac{1}{2}\sqrt{\frac{3\hbar\tau}{2mL^2}}e^{\frac{t}{\tau}}$ or $e^{\frac{t}{\tau}} \leq \sqrt{\frac{2mL^2}{3\hbar\tau}}$ , i.e.,

$$t \leq \tau \ln\left(\sqrt{\frac{2mL^2}{3\hbar\tau}}\right) = \sqrt{\frac{3L}{2g}} \ln\left(\sqrt{\frac{2mL^2}{3\hbar\sqrt{\frac{3L}{g}}}}\right) = \sqrt{\frac{3L}{g}} \ln\sqrt{\frac{2mL^{3/2}g^{1/2}}{3\hbar 3^{1/2}}}$$

$$t \leq \sqrt{\frac{3L}{2g}} \ln\left(\frac{4m^2L^3g}{\hbar^2 3^6}\right)^{\frac{1}{4}}$$

$$t \leq \frac{1}{4}\sqrt{\frac{3L}{2g}} \ln\left(\frac{4m^2L^3g}{3^6\hbar^2}\right)$$

Putting in typical numbers, *m* = 10 g, *L* = 10 cm we get $t \leq \frac{1}{4} \times 0.12 \times \ln(4.9 \times 10^{59}) \approx 4$ s . No matter how hard we try we will never get to balance the pencil for more than 4 s!

Quantum physics limits the time to just 4 s, down from an infinite time that would be expected from classical physics alone.

Obviously, this is a very rough estimate. We used the approximation $\sin\theta \approx \theta$ throughout which is only valid when the angle is very small.

It is quite extraordinary that a quantum mechanical principle manifests itself in this macroscopic situation. In principle, one could use this to estimate Planck's constant: you try to balance the pencil and time its fall. This will give an estimate of the logarithmic factor and hence Planck's constant. But you need some sort of machine that will position the pencil as vertically as possible, something that cannot be done by hand.